WAVELET ANALYSIS FOR ANDERSON WAVEFUNCTIONS IN ONE AND TWO DIMENSIONS

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We study the local scaling behavior of amplitude fluctuations of localized Anderson wavefunctions in one and two dimensions. The corresponding measures are analysed by means of the standard multifractal method and by a recently introduced wavelet approach to multifractality. Our calculations suggest that such localized wavefunctions are not multifractal in one dimensional space, and have similar statistical properties as profiles generated by simple random walks. In two dimensions, strong finite size effects prevent a precise determination of the corresponding exponents. In this case, the wavelet approach becomes an essential tool for elucidating the actual behavior of the measure, suggesting that amplitude fluctuations are also regular in two dimensions.

1 Introduction

In recent years, much attention has been drawn to the study and characterization of random distributions displaying self-similar fluctuations in some range of length scales. Prominent examples are the spatial distribution of energy dissipation in turbulence [1–3], the phase space structure of strange attractors in chaotic dynamical systems [4], the strong amplitude fluctuations of electronic wavefunctions at the metal-insulator transition in disordered systems [5], and many other physical systems [6,7].

To fully characterize self-similar fluctuations an infinite number of fractal dimensions is required [1,2]. The corresponding distributions are called multifractal, and can be quantitatively analysed with the so-called box-counting method (BCM), commonly referred to as multifractal analysis [8,9].

In this work, we consider measures corresponding to localized Anderson wavefunctions in one and two-dimensional disordered systems (Anderson measures). For these systems, the scaling behavior of the measure is not known exactly and the BCM has led to controversial interpretations [10–13]. Recently, we have discussed an alternative method, based on a discrete wavelet approach to multifractality [14], showing unambiguously that Anderson measures are not multifractal in one-dimensional space [15]. Here, we extend our previous studies to the more difficult two dimensional case, and show that conclusive results can not be obtained by applying the BCM alone. In this case, the wavelet analysis becomes essential to discern the correct asymptotic behavior of the measure.

The paper is organized as follows. In Sec. 2 we briefly review the results for Anderson wavefunctions in one dimension, obtained with the BCM and the wavelet approach. In Sec. 3 we consider the two dimensional case, and discuss the possible scenarios of the measure as obtained from the two methods. Finally, in Sec. 4, we present our concluding remarks.

2 Anderson model in one-dimension

In the Anderson model, the wave-function ψ_i at the lattice site i obeys the one-band tight-binding equation,

$$t_{i,i-1}\ \psi_{i-1} + t_{i,i+1}\ \psi_{i+1} + \epsilon_i\ \psi_i = E\ \psi_i \tag{1}$$

where $t_{i,i\pm1} = -1$ are the off-diagonal matrix elements, ϵ_i the diagonal matrix elements, and E is the eigenvalue. Here, periodic boundary conditions are used to solve Eq. 1. The site energies ϵ_i are taken to be uncorrelated random numbers uniformly distributed in the interval $-w/2 < \epsilon < w/2$ (diagonal disorder), where w denotes the amplitude of disorder. The wavefunctions are normalized according to $\sum_i |\psi_i|^2 = 1$.

It is known that $|\psi_i|^2$ displays strong amplitude fluctuations and it is localized around the localization center, i_c, upon averaging over many wave-functions with energies having a width δE around E, as

$$\langle|\psi_{j+i_c}|^2\rangle \sim \exp(-2|j|/\lambda)$$

with a localization length [16] $\lambda(w) \cong 105/w^2$ for $E = 0$.

2.1 Multifractal analysis

Let us proceed by applying the multifractal analysis to Anderson wavefunctions. We define a measure $\phi(x)$ (≥ 0) corresponding to ψ_i as $\phi(i) = |\psi_i|^2$, and denote it for short the Anderson measure. Here, the index i runs over all lattice sites, $0 \leq i < L$, where L is the lattice length, and the lattice constant is taken equal to one. Then, the corresponding box-probabilities are obtained as

$$p(x,\ell) = \sum_{x \leq i' \leq x+\ell-1} \phi(i'),$$

where $x \in \{0, \ell, 2\ell, 3\ell, ...\}$, and we use $\ell = 2^n$ with $n \geq 0$. To describe the scaling behavior of the measure, one studies the generalized moments (or partition function)

$$Z_q(\ell) \ = \ \sum_x \ \langle p^q(x,\ell)\rangle \ \sim \ \ell^{\tau(q)}, \tag{2}$$

averaged over many wavefunctions with eigenvalues E, as a function of length scale ℓ. For small ℓ ($\ell \ll \lambda$ in our case), Eq. 2 defines the scaling exponents $\tau(q)$. For a regular, i.e. non-multifractal measure $\tau(q) = d\ (q-1)$, where d is the dimension of the support of the measure. On the contrary, for a multifractal measure, $\tau(q)$ becomes a non-linear function of q.

Let us now calculate Z_q for Anderson measures in $d = 1$. Results for $Z_2(\ell)$, were obtained for different amplitudes of disorder w, ranging from $w = 0.125$ to $w = 1$. To determine the exponents $\tau(2)$, we have calculated the local slopes of $Z_2(\ell)$ versus ℓ, defined as

$$\tau(2,\ell) = \frac{\log Z_2(2\ell) - \log Z_2(\ell)}{\log 2} \tag{3}$$

and show them in the inset of Fig. 1. The local slopes tend to the asymptotic value $\tau(2) = 1$ for $\ell \to 1$, indicating regular behavior. The observed dependence of $\tau(2,\ell)$ on the disorder parameter w is due to localization effects governed by the finite localization length λ: The smaller the value of λ, the smaller the range of length scales $\ell \ll \lambda$ within which the asymptotic behavior in Eq. 2 can be expected.

Thus, to take localization effects explicitely into account we make the ansatz

$$\tau(2,\ell) = \tau(2) - \mathrm{A}\ \left(\frac{\ell}{\lambda}\right)^{\gamma} \tag{4}$$

containing three fit parameters, $\tau(2)$, A and γ. The excellent data collapse, shown in Fig. 1, supports our ansatz Eq. 4. The best fit to the data yields $\tau(2) = 1.001 \pm 0.002$, $\mathrm{A} = 0.54 \pm 0.10$ and $\gamma = 0.77 \pm 0.10$. These results further suggest that $\tau(2) = 1$, i.e. Anderson measures are non-multifractal in one-dimensional space.

2.2 Wavelet analysis

The previously found scaling behavior of Anderson measures in one dimension can be highlighted with a discrete wavelet analysis, referred to as discrete wavelet approach (DWA) to multifractality [14,15]. We briefly review the DWA in the following.

The DWA starts by calculating the "differential" partition function

$$W_q(\ell) \ = \ \sum_x \ \langle |p(x,\ell) - p(x+\ell,\ell)|^q \rangle \ \sim \ \ell^{\beta(q)} \tag{5}$$

where now $x \in \{0, 2\ell, 4\ell, 8\ell, ...\}$, and the scaling exponents $\beta(q)$ play a role analogous to $\tau(q)$ in Eq. 2. For a multifractal measure, one can show that $\beta(q) = \tau(q)$ under quite general conditions [14].

Figure 1: Plot of the local slopes $\tau(2,\ell)$ versus $\ell/\lambda(w)$, for eigenvalues $E \cong 0$ ($\delta E \cong 0.0001$). The symbols correspond to four different amplitudes of disorder: $w = 0.125$ (circles), 0.25 (squares), 0.5 (stars), and 1 (triangles), for lattices of length $L = 2^{18}$, 2^{16}, 2^{14}, and 2^{12}, respectively. Averages over 10^4 wavefunctions each were performed. The line is the best non-linear fit to the form $\tau(2,\ell) = \tau(2) - \mathrm{A}\ (\ell/\lambda)^{\gamma}$, with $\tau(2) = 1.001 \pm 0.002$, $\mathrm{A} = 0.54 \pm 0.10$ and $\gamma = 0.77 \pm 0.10$. In the inset, the slopes $\tau(2,\ell)$ are plotted versus ℓ, and the curves are guides to the eye.

Figure 2: Plot of the local slopes $\beta(2,\ell)$ versus $\ell/\lambda(w)$, for the same wavefunctions considered in Fig. 1. The line is the best non-linear fit to the form $\beta(2,\ell) = \beta(2) - \mathrm{A}\ (\ell/\lambda)^{\gamma}$, with $\beta(2) = 2.03 \pm 0.05$, $\mathrm{A} = 1.02 \pm 0.10$ and $\gamma = 0.43 \pm 0.10$. In the inset, the slopes $\beta(2,\ell)$ are plotted versus ℓ, and the curves are guides to the eye.

For a regular measure, however, the differences between box probabilities contain additional information described by an exponent H, characteristic of the measure. It can be shown that $\langle |p(x,\ell) - p(x+\ell,\ell)|^q \rangle \sim \ell^{qH} \langle p^q(x,\ell) \rangle$ [14]. Then, Eq. 5 becomes

$$W_q(\ell) \sim \ell^{qH} \; Z_q(\ell) \sim \ell^{qH+\tau(q)} \tag{6}$$

where $\tau(q) = d\;(q-1)$, and

$$\beta(q) = \tau(q) + H\; q \tag{7}$$

differing from $\tau(q)$ by the additional term Hq, indicating absence of multifractal behavior in space.

In the following, we apply the DWA to the Anderson measures discussed above, and consider the case $q = 2$ again. ¿From $W_2(\ell)$ we calculate the local slopes $\beta(2,\ell)$, and repeat the analysis performed above for $\tau(2,\ell)$. As shown in the inset of Fig. 2, $\beta(2,\ell)$ fluctuates quite strongly as a function of ℓ, more than $\tau(2,\ell)$ does. However, one can still draw a conclusion regarding the asymptotic value $\beta(2)$. Indeed, the data suggest that $\beta(2) \cong 2$, by considering values $\ell \geq 8$. The deviations for $\ell \leq 4$ are due to the inherent property of the wavefunction ψ_i, which changes periodically from a large amplitude at one site to a small amplitude at its neighboring site. This property, which is revealed by the wavelet approach, can be discarded from our scaling analysis.

The value $\beta(2) \cong 2$ indicates that $H \cong 1/2$ according to Eq. 7, implying a regular behavior of the measure. As further shown in Fig. 2, it is found that $\beta(2,\ell)$ also obeys a scaling ansatz similar to Eq. 4, where now $\beta(2) = 2.03 \pm 0.05$, A $= 1.02 \pm 0.10$ and $\gamma = 0.43 \pm 0.10$. Since the value of $\beta(2)$ is found to be definitely larger than $\tau(2)$, we may conclude that Anderson measures are regular in one dimension. In addition, our results suggest that they are characterized by an exponent $H = 1/2$, as for profiles generated by simple random walks [14].

3 Anderson model in two-dimensions

Next, we consider the Anderson model in two dimensions. The corresponding TB equation now reads

$$-(\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1}) + \epsilon_{i,j}\; \psi_{i,j} = E\; \psi_{i,j} \tag{8}$$

where the site energies $\epsilon_{i,j}$ are randomly distributed in the interval $-w/2 < \epsilon < w/2$.

Figure 3: A wavefunction for the Anderson model in two dimension for disorder $w = 7$ and eigenvalue $E \cong 0$, on a square lattice of 100×100 sites.

It is known that also in $d = 2$ the wavefunctions are all localized and the localization lengths $\lambda(w)$ increase more rapidly than in $d = 1$ when the amplitude of disorder w vanishes [17]. For $w = 4$, for instance, $\lambda \cong 500$ which means that lattices of at least 500×500 sites are required to examine the localization of the wavefunctions appropriately. This would imply the diagonalization of a $2.5 \cdot 10^5 \times 2.5 \cdot 10^5$ matrix, which is an impracticable task considering that averages over several wavefunctions and realizations of disorder must be performed.

In our calculations, we have considered square lattices of 180×180 sites, and studied the cases $w = 7$, $w = 5.5$ and $w = 4$, corresponding to localization lengths $\lambda \cong 19$, 60 and 500, respectively [17]. The data for $w = 4$ is included for illustration since $\lambda(4)$ exceeds the lattice size. The wavefunctions were obtained by employing the Lanczos method with periodic boundary conditions.

As shown in Fig. 3, the wavefunctions display strong amplitude fluctuations in space, and the question is whether such fluctuations are characterized by a multifractal behavior of the measure or not. We have tried to answer this question here by performing both a BCM and a wavelet analysis. We start by discussing the first method.

3.1 Multifractal analysis

Similarly as in $d = 1$, we define the box-probabilities as

$$p(x, y, \ell) = \sum_{i,j} \phi(i, j),$$

where $x, y \in \{0, \ell, 2\ell, 3\ell ...\}$, for several values $\ell \in [1, 180]$, and the sum of the measure $\phi(i, j) = |\psi_{i,j}|^2$ is over all sites (i, j) within the box of length ℓ around (x, y). The generalized moments (or partition function) now reads

$$Z_q(\ell) \;=\; \sum_{x,y} \langle p^q(x, y, \ell)\rangle \tag{9}$$

and the average is performed over many wavefunctions with eigenvalues E. Again, the scaling behavior of $Z_q \sim \ell^{\tau(q)}$ as a function of length scale ℓ is expected here for $\ell \ll \lambda$.

Results for $Z_2(\ell)$, and the corresponding local slopes $\tau(2, \ell)$ as defined in Eq. 3, are shown in Fig. 4a and b. As one can see from Fig. 4a, an apparent slope $\tau_{\rm eff}(2) = 1.6$ fits the data quite well for small ℓ/λ [a]. The local slopes $\tau(2, \ell)$, shown in Fig. 4b, indicate that such an effective slope can not correspond to the true asymptotic value. However, from these results one can not discard the possibility of a multifractal behavior of the measure. In the hope to shed some light on the scaling behavior of these wavefunctions we apply the wavelet analysis next.

3.2 Wavelet analysis

For the DWA in two dimensions, there are three possible definitions for the W_q-partition functions [14]. One of those functions is [b],

$$W_q(\ell) = \sum_{x,y} \langle |p(x, y, \ell) - p(x, y + \ell, \ell) + p(x + \ell, y, \ell) - p(x + \ell, y + \ell, \ell)|^q\rangle \tag{10}$$

where $x, y \in \{0, 2\ell, 4\ell ...\}$. As in $d = 1$, we expect that $W_q(\ell) \sim \ell^{\beta(q)}$ in some appropriate range of length scales $\ell \ll \lambda$ also here.

Results for $W_2(\ell)$, shown in Fig. 4c, indicate that again an effective slope $\beta_{\rm eff}(2) = 1.6$ fits the data well, suggesting multifractal behavior. Note that

[a] The deviations of the points from the data collapse for large ℓ/λ are due to our relatively small lattice size used.

[b] The other two possible definitions of W_q yield similar results and will not be discussed here.

Figure 4: Multifractal and wavelet analysis in $d = 2$ as a function of ℓ/λ for wavefunctions with $E \cong 0$ on square lattices of 180×180 sites. The symbols correspond to different values of disorder: $w = 7$ (full squares), 5.5 (full triangles) and 4 (open circles). Averages over 700 configurations were performed for the first two cases, and over 350 for the latter. Here we use the values $\lambda = 19$, 60 and 130, respectively, to obtain a data collapse at small ℓ/λ. (a) The partition function $Z_2(\ell)$. The continuous line is a fit to the data and has slope $\tau_{\rm eff}(2) = 1.6$. The dashed line has slope 2 and is shown for illustration. (b) The local slopes $\tau(2,\ell)$ for the same data shown in (a). The continuous line is a fit with the form: $\tau(2,\ell) = \tau(2) - \mathrm{A}\ (\ell/\lambda)^\gamma$, with $\tau(2) = 2.0 \pm 0.3$, $\mathrm{A} = 1.2 \pm 0.3$ and $\gamma = 0.3 \pm 0.1$. The dashed line corresponds to the value $\tau_{\rm eff}(2) = 1.6$ as obtained in (a). (c) The function $W_2(\ell)$. The continuous line is a fit to the data and has slope $\beta_{\rm eff}(2) = 1.6$. The dashed line has slope 3 and is shown for illustration. (d) The local slopes $\beta(2,\ell)$ for the same data shown in (c). The continuous line is a fit with the form: $\beta(2,\ell) = \beta(2) - \mathrm{A}\ (\ell/\lambda)^\gamma$, with $\beta(2) = 2.5 \pm 0.5$, $\mathrm{A} = 1.4 \pm 0.5$ and $\gamma = 0.25 \pm 0.20$. The dashed line corresponds to the value $\beta_{\rm eff}(2) = 1.6$ as obtained in (c).

the quality of the data collapse has improved now with respect to $Z_2(\ell)$. One can also note that the hypothetical value $\beta(2) = 3$, corresponding to a regular measure with $H = 1/2$ (see Eq. 7) and represented by the dashed line in the figure, seems to be inconsistent with the present data. Different is the situation by looking at the successive slopes $\beta(2,\ell)$ as shown in Fig. 4d. A fit of the numerical results with the form $\beta(2,\ell) = \beta(2) - \mathrm{A}\ (\ell/\lambda)^\gamma$, suggests that $\beta(2) > 2$, providing us with a substantial evidence that the measure may be regular asymptotically.

4 Concluding remarks

In summary, we have applied a discrete wavelet approach for analysing local scaling behavior of Anderson localized wavefunctions in one and two dimensions. We have shown that the corresponding measures are non-multifractal in one dimension, and are characterized by the exponent $H = 1/2$ like profiles generated by simple random walks. In two dimensions, the results are much less conclusive than in $d = 1$ because the numerical calculations are necessarily limited to the strong disorder case where the localization lengths are smaller or at most comparable to the available lattice sizes. Yet, the results obtained from the wavelet analysis indicate that Anderson measures may be regular in two dimensions, too. In this case, although the characteristic exponent H could not be determined accurately, it is still consistent with the value $H = 1/2$ within our error bars. Further work to clarify this point is therefore highly desirable.

Acknowledgments

We have benefitted from discussions with M. Greiner, B. Kramer, and L. Schweitzer.

References

1. B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
2. U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil *et al.*, (North-Holland, Amsterdam, 1985), pp. 84-88.
3. C. Menevean and K.R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
4. P. Grassberger and I. Procaccia, Physica D **13**, 34 (1984).
5. C. Castellani and L. Peliti, J. Phys. A **19**, L429 (1986); C. Castellani, C. Di Castro and L. Peliti, J. Phys. A **19**, L1099 (1986).
6. A. Bunde and S. Havlin, eds., *Fractals and Disordered Systems* (2nd edition, Springer, Heidelberg, 1996).
7. L. Pietronero and E. Tosatti, eds., *Fractals in Physics* (North Holland, Amsterdam, 1986).
8. T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
9. J. Feder, *Fractals* (Plenum, New York, 1988).
10. C.M. Soukoulis and E.N. Economou, Phys. Rev. Lett. **52**, 565 (1984).
11. L. Pietronero, A.P. Siebesma, E. Tosatti, and M. Zanetti, Phys. Rev. B **36**, 5635 (1987).
12. H.E. Roman, Phys. Rev. Lett. **68**, 2856 (1992).
13. B. Huckestein and L. Schweitzer, J. Non-Crystalline Solids **164**, 461 (1993).
14. J.W. Kantelhardt, H.E. Roman and M. Greiner, Physica A **220**, 219 (1995).
15. H.E. Roman, J.W. Kantelhardt and M. Greiner, Europhys. Lett. (in press).
16. E.N. Economou, *Green's Functions in Quantum Physics*, (Springer, Heidelberg, 1983), p. 173.
17. A. MacKinnon and B. Kramer, Z. Phys. B **53**, 1 (1983).